

THE DE RHAM COHOMOLOGY OF SUBCARTESIAN SPACES

CHARLES D. MARSHALL

The notion of differentiable subcartesian space is a generalization of that of differentiable manifold. Arbitrary subsets of \mathbf{R}^n are special examples as well as differentiable manifolds with boundary, or corners, and analytic or semi-analytic spaces. In [7] we constructed the category of C^∞ -subcartesian spaces and introduced the calculus of tensor fields and differential forms. In this sequel to [7] we study the cohomology algebra formed from those differential forms.

In § 1 we define the de Rham cohomology of a C^∞ -subcartesian space. In § 2 we establish the Eilenberg-Steenrod axioms on an appropriate admissible category of pairs of subcartesian spaces. In § 3 we show by example that the de Rham and Čech cohomologies are distinct. We then establish a spectral sequence which has its E_2 -terms in sheaf cohomology and which converges in the de Rham cohomology. We introduce a graded-sheaf invariant $\mathcal{H}(S)$ of a differentiable subcartesian space S , the *de Rham sheaf* of S , whose vanishing in higher degrees is sufficient for the de Rham cohomology to be naturally isomorphic to the sheaf cohomology with coefficients in $\mathcal{H}^0(S)$. If S is locally contractible, then $\mathcal{H}^0(S) = \mathbf{R}$ and $\mathcal{H}^k(S) = 0$ for $k > 0$, thus giving a natural isomorphism of the de Rham and sheaf-theoretic cohomology theories. We finish with an appendix on the C^k -cohomology, showing that it is not a topological invariant.

It is perhaps worth while to compare the cohomology theory developed here with those of [10], [11], and [12]. In [10] Schwartz constructed a cohomology theory which coincides with Čech cohomology on finite dimensional compact spaces. Example 3.1 shows that this is not always the case for our theory. In [11] Smith constructed an exterior differential algebra and cohomology theory for each pair (X, \mathcal{F}) , where X is a topological space and \mathcal{F} is a set of continuous \mathbf{R} -valued functions on X . One might expect our theory to follow as a special case of Smith's when X is a C^∞ -subcartesian space and $\mathcal{F} = C^\infty(X)$, but Example 3.15 shows that this is not the case. In [12] Spallek considered several notions of differential forms on differentiable spaces and stated a de Rham isomorphism theorem. In [7] we showed that the differential forms as defined for subcartesian spaces and the differential forms of [12] are different. Whether

the corresponding cohomology theories are isomorphic is an open question.

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1. Definition of the de Rham cohomology

For each C^∞ -subcartesian spaces S we shall denote the graded $C^\infty(S)$ -algebra of alternating covariant tensors (also called *forms*) by $F(S) = \{F^k(S) | k \in \mathbf{Z}\}$, the subalgebra of forms having differentials by $D(S) = \{D^k(S) | k \in \mathbf{Z}\}$, the graded ideal of differentials of 0 by $m(S) = \{m^k(S) | k \in \mathbf{Z}\}$, and the graded algebra of differential forms by $A(S) = \{A^k(S) | k \in \mathbf{Z}\}$. All of the homogeneous submodules of grades $k < 0$ are 0 by definition. If $f: S \rightarrow R$ is a C^∞ -mapping of subcartesian spaces, then $f^*: F(R) \rightarrow F(S)$ maps $D(R) \rightarrow D(S)$, $m(R) \rightarrow m(S)$, and hence induces the pull-back $A(R) \rightarrow A(S)$ (also denoted f^*). Thus the following systems with their corresponding systems of pull-backs of inclusions are presheaves of modules over the presheaf C_s^∞ of C^∞ -functions:

$$(1.1) \quad \begin{aligned} F_S &:= \{F(U) | U \text{ open in } S\}, & D_S &:= \{D(U) | U \text{ open in } S\}, \\ m_S &:= \{m(U) | U \text{ open in } S\}, & A_S &:= \{A(U) | U \text{ open in } S\}. \end{aligned}$$

All four satisfy the sheaf axiom F_1 of Godement [5] for arbitrary S , and F_S satisfies sheaf axiom F_2 . If S is paracompact, then all four satisfy both F_1 and F_2 . In any case we denote the corresponding generated sheaves (*espaces etales*) by $\mathcal{F}_S, \mathcal{D}_S, \mathcal{M}_S$ and \mathcal{A}_S . Note that $\mathcal{F}_S = \mathcal{D}_S$ and $\mathcal{A}_S = \mathcal{F}_S / \mathcal{M}_S$.

Since exterior differentiation commutes with pull-backs [7], (\mathcal{A}_S, d) is a differential graded presheaf, and (\mathcal{A}_S, d) is a differential graded sheaf. A C^∞ -mapping $f: S \rightarrow S'$ induces f -cohomorphisms (cf. Bredon [3]) $F_{S'} \rightarrow F_S, D_{S'} \rightarrow D_S, m_{S'} \rightarrow m_S, A_{S'} \rightarrow A_S$ and hence an f -cohomorphism $f^*: \mathcal{A}_{S'} \rightarrow \mathcal{A}_S$. Each of these is compatible with differentiation.

Lemma 1.2. *Let Σ be a paracompactifying family of supports on S , and let \mathcal{S} be a sheaf of R -vector spaces over S . Then $\mathcal{F}_S \otimes \mathcal{S}, \mathcal{M}_S \otimes \mathcal{S}$, and $\mathcal{A}_S \otimes \mathcal{S}$ are Σ -soft and Σ -fine.*

Proof. Existence of C^∞ -partitions of unity on paracompact subcartesian spaces [7, Proposition 1.2], and Σ being paracompactifying imply that \mathcal{F}_S^0 is Σ -fine and Σ -soft. Since $\mathcal{F}_S \otimes \mathcal{S}, \mathcal{M}_S \otimes \mathcal{S}$, and $\mathcal{A}_S \otimes \mathcal{S}$ are \mathcal{F}_S^0 -modules, it follows that each is Σ -soft and Σ -fine. q.e.d.

Let $i: R \subset S$, and let \mathcal{S} be a sheaf of R -vector spaces over S . Define $\mathcal{H}(S, R; \mathcal{S})$ to be the sheaf of germs of local sections γ of $\mathcal{A}_S \otimes_R \mathcal{S}$ such that $(i^* \otimes |_R)\gamma = 0$ (where $|_R$ is the restriction cohomomorphism $\mathcal{S} \rightarrow \mathcal{S}|_R$). Equivalently, $\mathcal{H}(S, R; \mathcal{S})$ is the kernel of the unique homomorphism $j: \mathcal{A}_S \otimes \mathcal{S} \rightarrow i(\mathcal{A}_R \otimes \mathcal{S}|_R)$ such that $i^* \otimes |_R$ has the factorization $\mathcal{A}_S \otimes \mathcal{S} \rightarrow i(\mathcal{A}_R \otimes \mathcal{S}|_R) \rightarrow \mathcal{A}_R \otimes \mathcal{S}|_R$ (cf. [3, p. 9ff]). An elementary argument

shows that $\mathcal{K}(S, R; \mathcal{S}) = \mathcal{K}(S, R) \otimes \mathcal{S}$, (where $\mathcal{K}(S, R) := \mathcal{K}(S, R; R)$) when $R \subseteq S$ is closed.

Define $\delta := d \otimes_R \text{Id} : \mathcal{A}_S \otimes \mathcal{S} \rightarrow \mathcal{A}_S \otimes \mathcal{S}$. Then $\delta^2 = 0$ and δ leaves $\mathcal{K}(S, R; \mathcal{S})$ invariant. Thus $(\mathcal{A}_S \otimes \mathcal{S}, \delta)$ and $(\mathcal{K}(S, R; \mathcal{S}), \delta)$ are differential graded sheaves. If Σ is a family of supports on S , then the following sequence of complexes of R -vector spaces is exact:

$$(1.3) \quad 0 \rightarrow \Gamma_\Sigma \mathcal{K}(S, R; \mathcal{S}) \rightarrow \Gamma_\Sigma \mathcal{A}_S \otimes \mathcal{S} \rightarrow \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes (\mathcal{S}|_R).$$

Definition 1.4. The complex $\Gamma_\Sigma \mathcal{K}(S, R; \mathcal{S}) = : K_\Sigma(S, R; \mathcal{S})$ is called the de Rham complex of (S, R) with coefficients \mathcal{S} and supports Σ . We define the de Rham cohomology of (S, R) with coefficients \mathcal{S} and supports Σ to be the homology of $K_\Sigma(S, R; \mathcal{S})$, that is,

$$H_\Sigma^k(S, R; \mathcal{S}) := H^k K_\Sigma(S, R; \mathcal{S}), \quad k \in \mathbb{Z}.$$

Obviously,

$$(1.5) \quad H_\Sigma^k(S, R; \mathcal{S}) = 0 \quad \text{for all } k < 0.$$

When S is paracompact, $R = \emptyset$, $\Sigma = \text{cls}$ (all closed subsets of S), and $\mathcal{S} = R$, then $K_\Sigma(S, R; \mathcal{S})$ is naturally isomorphic to the complex of differential forms $A(S)$, and

$$H^k(S) := H_{\text{cls}}^k(S, \emptyset; R) \approx \frac{\text{Ker } d : A^k(S) \rightarrow A^{k+1}(S)}{\text{Im } d : A^{k-1}(S) \rightarrow A^k(S)},$$

i.e., *closed differential k -forms modulo exact differential k -forms*. Moreover, if we define a k -form $\zeta \in F^k(S)$ to be *closed* when $0 \in F^{k+1}(S)$ is one of its differentials, and *exact* when it is a differential of some $\omega \in F^{k-1}(S)$, then

$$H^k(S) \approx \frac{\text{closed } k\text{-forms}/\mathfrak{m}^k(S)}{\text{exact } k\text{-forms}/\mathfrak{m}^k(S)} \approx \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$$

Similarly, if S and R are arbitrary, Σ is paracompactifying for the pair (S, R) (cf. [3]), and $\mathcal{S} = R$, then

$$\begin{aligned} H_\Sigma^k(S, R) &:= H_\Sigma^k(S, R; R) \\ &\approx \frac{\{\omega \in A^k(S) \mid \text{supp } \omega \in \Sigma \cap (S \setminus R) \text{ and } d\omega = 0\}}{\{\omega \in A^k(S) \mid \exists \zeta \in A^{k-1}(S) \text{ such that } \text{supp } \zeta \in \Sigma \cap (S \setminus R) \text{ and } d\zeta = \omega\}} \\ &\approx \frac{\{\varphi \in F^k(S) \mid \text{supp } \varphi \in \Sigma \cap (S \setminus R), \text{ and } 0 \in d\varphi\}}{\{\varphi \in F^k(S) \mid \exists \theta \in F^{k-1}(S) \text{ with } \varphi \in d\theta \text{ and } \text{supp } \theta \in \Sigma \cap (S \setminus R)\}}, \end{aligned}$$

i.e., *k -forms closed relative to R modulo k -forms exact relative to R* .

Let R', \mathcal{S}' , and Σ' be alternate choices of R, \mathcal{S} and Σ . Define

$$(1.6) \quad \begin{aligned} \wedge : (\mathcal{A}_S \otimes \mathcal{S}) \otimes (\mathcal{A}_S \otimes \mathcal{S}') &\rightarrow \mathcal{A}_S \otimes \mathcal{S} \otimes \mathcal{S}'; \\ (\alpha \otimes \sigma) \wedge (\alpha' \otimes \sigma') &:= \alpha \wedge \alpha' \otimes \sigma \otimes \sigma'. \end{aligned}$$

This product determines a product

$$(1.7) \quad \wedge : K_{\Sigma}(S, R; \mathcal{S}) \otimes K_{\Sigma'}(S, R'; \mathcal{S}') \rightarrow K_{\Sigma \cap \Sigma'}(S, R \cup R'; \mathcal{S} \otimes \mathcal{S}').$$

Products of cocycles are cocycles, and the product of a cocycle and coboundary is a coboundary. Thus \wedge induces a product

$$\cup : H_{\Sigma}^k(S, R; \mathcal{S}) \otimes H_{\Sigma'}^m(S, R'; \mathcal{S}') \rightarrow H_{\Sigma \cap \Sigma'}^{k+m}(S, R \cup R'; \mathcal{S} \otimes \mathcal{S}'), k, m \in \mathbb{Z}.$$

2. The Eilenberg-Steenrod axioms

Let (S, R) and (S', R') be pairs of subcartesian spaces, Σ and Σ' families of supports on (S, R) and (S', R') , and let \mathcal{S} and \mathcal{S}' be sheaves of \mathbf{R} -vector spaces over S and S' . Let $f: (S, R) \rightarrow (S', R')$ be a C^∞ -mapping of pairs, proper with respect to Σ and Σ' (i.e., $f^{-1}(F) \in \Sigma$ for each $F \in \Sigma'$). Let $g: \mathcal{S}' \rightarrow \mathcal{S}$ be an f -cohomomorphism. Then the f -cohomomorphism $f^* \otimes g: \mathcal{A}_{S'} \otimes \mathcal{S}' \rightarrow \mathcal{A}_S \otimes \mathcal{S}$ induces a homomorphism of complexes $K_{\Sigma'}(S', R'; \mathcal{S}') \rightarrow K_{\Sigma}(S, R; \mathcal{S})$ and hence homomorphisms

$$(f, g)_\#^k : H_{\Sigma'}^k(S', R'; \mathcal{S}') \rightarrow H_{\Sigma}^k(S, R; \mathcal{S}), \quad k \in \mathbb{Z}.$$

Let \mathcal{Q} be the category whose objects are quadruples $(S, R, \mathcal{S}, \Sigma)$ and whose morphisms are pairs $(f, g): (S, R, \mathcal{S}, \Sigma) \rightarrow (S', R', \mathcal{S}', \Sigma')$, where $f: (S, R) \rightarrow (S', R')$ is C^∞ and proper, and $g: \mathcal{S}' \rightarrow \mathcal{S}$ is an f -cohomomorphism. Then $(H, \#)$ is a contravariant functor from \mathcal{Q} to the category of graded \mathbf{R} -vector spaces. The induced homomorphism $(f, g)_\#$ is compatible with \cup -products.

If $u = (f, g): (S, R, \mathcal{S}, \Sigma) \rightarrow (S', R', \mathcal{S}', \Sigma')$ is a morphism in \mathcal{Q} , we shall write u^* for $f^* \otimes g: \mathcal{A}_{S'} \otimes \mathcal{S}' \rightarrow \mathcal{A}_S \otimes \mathcal{S}$. By abuse of notation, we may also write u^* for $\Gamma u^*: \Gamma_{\Sigma'} \mathcal{A}_{S'} \otimes \mathcal{S}' \rightarrow \Gamma_{\Sigma} \mathcal{A}_S \otimes \mathcal{S}$. If $i: R \subset S$ is an inclusion, then we shall write simply i^* for $i^* \otimes |_{\mathbf{R}}$, and $i^\#$ for the homomorphism in cohomology induced by i^* .

A homomorphism $f: \mathcal{S} \rightarrow \mathcal{S}'$ of sheaves over S is nothing but an id_S -cohomomorphism. In this special case we shall denote the induced homomorphism $(\text{id}_S, f)_\# : H_{\Sigma}(S, R; \mathcal{S}) \rightarrow H_{\Sigma}(S, R; \mathcal{S}')$ simply by $f_\#$. For each S, R and Σ the covariant functor $(H_{\Sigma}(S, R; \cdot), \#)$ is additive (in fact, strongly additive).

Theorem 2.1. *Let F^k denote the functor $(H_{\Sigma}^k(S, R; \cdot), \#)$, and Σ be a family of supports on S paracompactifying for the pair (S, R) . Then for each short exact sequence of sheaves of \mathbf{R} -vector spaces over S*

$$(2.2) \quad 0 \longrightarrow \mathcal{S}' \xrightarrow{f} \mathcal{S} \xrightarrow{g} \mathcal{S}'' \longrightarrow 0$$

and each $k \in \mathbf{Z}$, there is a homomorphism $b^k: F^k(\mathcal{S}'') \rightarrow F^{k+1}(\mathcal{S}')$ such that the cohomology sequence

$$\dots \longrightarrow F^k(\mathcal{S}') \xrightarrow{f^\sharp} F^k(\mathcal{S}) \xrightarrow{g^\sharp} F^k(\mathcal{S}'') \xrightarrow{b^k} F^{k+1}(\mathcal{S}') \longrightarrow \dots$$

is exact. Moreover, each b^k is natural, i.e., short commutative ladder diagrams yield long commutative ladder diagrams.

Proof. Tensoring (2.2) with $\mathcal{A}_S \otimes_R$ and applying Γ_Σ give the exact sequence of complexes

$$(2.3) \quad 0 \rightarrow \Gamma_\Sigma \mathcal{A}_S \otimes \mathcal{S}' \rightarrow \Gamma_\Sigma \mathcal{A}_S \otimes \mathcal{S} \rightarrow \Gamma_\Sigma \mathcal{A}_S \otimes \mathcal{S}'' \rightarrow 0$$

Since $\mathcal{A}_S \otimes \mathcal{S}'$ is Σ -soft, (2.3) remains exact when augmented on the right by zero. Similarly, the sequence

$$(2.4) \quad 0 \rightarrow \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes \mathcal{S}'|_R \rightarrow \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes \mathcal{S}|_R \rightarrow \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes \mathcal{S}''|_R \rightarrow 0$$

is exact. Applying the 3×3 lemma three times to the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_\Sigma(S, R; \mathcal{S}') & \rightarrow & K_\Sigma(S, R; \mathcal{S}) & \rightarrow & K_\Sigma(S, R; \mathcal{S}'') \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \Gamma_\Sigma \mathcal{A}_S \otimes \mathcal{S}' & \rightarrow & \Gamma_\Sigma \mathcal{A}_S \otimes \mathcal{S} & \rightarrow & \Gamma_\Sigma \mathcal{A}_S \otimes \mathcal{S}'' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes \mathcal{S}'|_R & \rightarrow & \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes \mathcal{S}|_R & \rightarrow & \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes \mathcal{S}''|_R \rightarrow 0 \end{array}$$

yields the exactness of the top row. The theorem now follows from the usual diagram chase (Snake lemma).

Theorem 2.5. *Let $(S, R, \mathcal{S}, \Sigma) \in \mathcal{Q}$ with R closed and Σ paracompactifying. Then there exist homomorphisms*

$$\Delta^k: H_{\Sigma \cap R}^k(R; \mathcal{S}|_R) \rightarrow H_\Sigma^{k+1}(S, R; \mathcal{S})$$

making the cohomology sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_\Sigma^k(S, R; \mathcal{S}) & \longrightarrow & H_\Sigma^k(S; \mathcal{S}) & \longrightarrow & H_{\Sigma \cap R}^k(R; \mathcal{S}|_R) \\ & & & & & & \xrightarrow{\Delta^k} \\ & & & & & & H_\Sigma^{k+1}(S, R; \mathcal{S}) \longrightarrow \dots \end{array}$$

exact. Each Δ^k is natural, i.e., if $f: (S, R, \mathcal{S}, \Sigma) \rightarrow (S', R', \mathcal{S}', \Sigma')$ is a morphism in \mathcal{Q} , R' is closed, and Σ' is paracompactifying, then $\Delta^k \circ (f|_R)^\# = f^\# \circ \Delta^k$. Given sequence (2.2), then $\Delta \circ b = b \circ \Delta = 0$.

Proof. Because \mathcal{A}_S is Σ -soft and $\mathcal{A}_{S'}$ is Σ' -soft, (1.3) remains exact when augmented on the right by zero. Thus we have the following commutative diagram of complexes with exact rows:

$$\begin{array}{ccccccc}
 0 \rightarrow & K_{\Sigma}(S, R; \mathcal{S}) & \rightarrow & \Gamma_{\Sigma} \mathcal{A}_S \otimes \mathcal{S} & \rightarrow & \Gamma_{\Sigma \cap R} \mathcal{A}_R \otimes \mathcal{S}|_R & \rightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 \rightarrow & K_{\Sigma'}(S', R'; \mathcal{S}') & \rightarrow & \Gamma_{\Sigma'} \mathcal{A}_{S'} \otimes \mathcal{S}' & \rightarrow & \Gamma_{\Sigma' \cap R'} \mathcal{A}_{R'} \otimes \mathcal{S}'|_{R'} & \rightarrow 0
 \end{array}$$

Existence and naturality of Δ^k now follow as usual.

If $\gamma \in \Gamma_{\Sigma \cap R} \mathcal{A}_R^k \otimes \mathcal{S}|_R$ satisfies $\delta\gamma = 0$, $[\gamma]_R$ denotes the cohomology class of γ in $H_{\Sigma \cap R}^k(\mathcal{R}; \mathcal{S}|_R)$, and $\gamma' \in \Gamma_{\Sigma} \mathcal{A}_S \otimes \mathcal{S}$ is any preimage of γ , then

$$(2.6a) \quad \Delta^k[\gamma]_R = [\delta\gamma']_{(S, R)}.$$

Similarly, if $\gamma'' \in K_{\Sigma}^k(S, R; \mathcal{S}'')$ satisfies $\delta\gamma'' = 0$, $\gamma \in K_{\Sigma}^k(S, R; \mathcal{S})$ satisfies $\text{Id} \otimes g(\gamma) = \gamma''$, and $\gamma' \in K_{\Sigma}^{k+1}(S, R; \mathcal{S}')$ satisfies $\text{Id} \otimes f(\gamma') = \delta\gamma$, then

$$(2.6b) \quad b^k[\gamma''] = [\gamma'].$$

It follows that $b \circ \Delta = \Delta \circ b = 0$.

Theorem 2.7. *Let $\mathcal{S} \otimes \mathcal{S}'$ and $\mathcal{S}' \otimes \mathcal{S}$, $\mathcal{S} \otimes (\mathcal{S}' \otimes \mathcal{S}'')$ and $(\mathcal{S} \otimes \mathcal{S}') \otimes \mathcal{S}''$ be identified respectively via the usual natural isomorphisms. Then the cup product is associative, graded-anticommutative and $H_{\text{cls}}^0(S)$ -bilinear. Moreover,*

$$\cup : H_{\Sigma}^k(S, R, \mathcal{S}) \otimes H_{\Sigma'}^m(S, R; \mathcal{S}') \rightarrow H_{\Sigma \cap \Sigma'}^{k+m}(S, R \cup R'; \mathcal{S} \otimes \mathcal{S}')$$

is a natural transformation of functors on \mathcal{L} satisfying the following conditions:

- (i) Let $0 \rightarrow \mathcal{T}' \xrightarrow{h} \mathcal{S} \xrightarrow{g} \mathcal{T}'' \rightarrow 0$ be an exact sequence of sheaves of \mathbf{R} -vector spaces over S , and let Σ be paracompactifying for the pair (S, R) . If $c \in H_{\Sigma}^k(S, R; \mathcal{S})$ and $c' \in H_{\Sigma'}^m(S, R'; \mathcal{T}'')$ then $b^m(c') \cup c = b^{m+k}(c' \cup c)$.
- (ii) Let $i: R \subset S$ be closed, let Σ and Σ' be paracompactifying support families on S , and let \mathcal{S} and \mathcal{S}' be sheaves of \mathbf{R} -vector space on S . If $c \in H_{\Sigma \cap R}^k(\mathcal{R}; \mathcal{S}|_R)$ and $c' \in H_{\Sigma'}^m(S; \mathcal{S}')$, then $\Delta c \cup c' = \Delta(c \cup i^*c')$.

Proof. Associativity, graded-commutativity and bilinearity hold at the chain level and hence in cohomology. Naturality of \cup with respect to induced maps has already been mentioned and is clear.

To establish (i), let $\eta'' \in \Gamma_{\Sigma'}(\mathcal{A}_S^m \otimes \mathcal{T}'')$ and $\gamma \in \Gamma_{\Sigma}(\mathcal{A}_S^k \otimes \mathcal{S})$ be representatives of c' and c , respectively. Let $\eta \in \Gamma_{\Sigma'}(\mathcal{A}_S^m \otimes \mathcal{T})$ be a preimage of η'' under $\Gamma(\text{id} \otimes g)$, and let $\eta' \in \Gamma_{\Sigma'}(\mathcal{A}_S^{m+1} \otimes \mathcal{T}')$ be a preimage of $\delta\eta$ under $\Gamma(\text{id} \otimes h)$. Then $\Gamma((\text{id} \otimes h) \otimes \text{id})(\eta' \wedge \gamma) = \delta\eta \wedge \gamma$, and $\eta' \wedge \gamma$ is a representative of $b^m c' \cup c$. On the other hand, $\Gamma((\text{id} \otimes g) \otimes \text{id})(\eta \wedge \gamma) = \eta'' \wedge \gamma$ is a representative of $c' \cup c$. Because $\delta\gamma = 0$, $\delta(\eta \wedge \gamma) = \delta\eta \wedge \gamma$. Thus $\eta' \wedge \gamma$ is also a representative of $b^{k+m}(c' \cup c)$. Therefore $b^m c' \cup c = b^{k+m}(c' \cup c)$.

To establish (ii), let η and γ be representatives of c and c' , respectively. If η' is any preimage of η under the induced map $\Gamma \mathcal{A}_S \otimes \mathcal{S} \rightarrow \Gamma \mathcal{A}_R \otimes \mathcal{S}|_R$, then $\delta(\eta' \wedge \gamma)$ is a representative of $\Delta(c \cup i^*c')$. Because $\delta\gamma = 0$, $\delta(\eta' \wedge \gamma) = \delta\eta' \wedge \gamma$, and $\delta\eta' \wedge \gamma$ is a representative of $\Delta c \cup c'$. It follows that $\Delta(c \cup i^*c') = \Delta c \cup c'$.

Theorem 2.8 (Excision). *Let $U \subseteq S$ satisfy $U \subseteq \text{Interior } R$, and let i denote the inclusion $(S \setminus U, R \setminus U) \subseteq (S, R)$. Then*

$$i^* : H_{\Sigma}(S, R; \mathcal{S}) \rightarrow H_{\Sigma \cap (S \setminus U)}(S \setminus U, R \setminus U; \mathcal{S}|_{S \setminus U})$$

is an isomorphism.

Proof. We shall show that

$$i^* : K_{\Sigma}(S, R; \mathcal{S}) \rightarrow K_{\Sigma \cap (S \setminus U)}(S \setminus U, R \setminus U; \mathcal{S}|_{S \setminus U})$$

is an isomorphism. Injectivity is trivial. To show surjectivity, let $\gamma \in K_{\Sigma \cap (S \setminus U)}(S \setminus U, R \setminus U; \mathcal{S}|_{S \setminus U})$, and let $p \in \partial U$. There is a neighborhood V of p in S such that $V \subseteq R$. For each $q \in V \cap (S \setminus U)$, $\gamma_q = 0$. It follows that γ may be extended by 0 to all of S to give a section $\gamma' \in K_{\Sigma}(S, R; \mathcal{S})$. Then $i^*(\gamma') = \gamma$.

Theorem 2.9 (Dimension). *Let $S = P$ be a one-point space, and V an R -vector space. Then*

$$H^k(P; V) = \begin{cases} V, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

The proof is trivial.

Definition 2.10. A homotopy $h = (f, g) : (S, R, \mathcal{S}, \Sigma) \times I \rightarrow (S', R', \mathcal{S}', \Sigma')$ in \mathcal{Q} is a C^∞ -homotopy of pairs f with f proper relative to $\Sigma \times I$ and Σ' , and an f -cohomomorphism $g : \mathcal{S}' \rightarrow \mathcal{S} \times I$.

Theorem 2.11 (Homotopy invariance). *Let \mathcal{S} and \mathcal{S}' be sheaves of R -vector spaces over S and S' , respectively, and let h be a homotopy in \mathcal{Q} as above. Suppose R is closed in S . Then*

$$h_0^* = h_1^* : H_{\Sigma}^k(S', R'; \mathcal{S}') \rightarrow H_{\Sigma}^k(S, R; \mathcal{S}), \quad k \in \mathbb{Z}.$$

Proof. For each $t \in I$ and $p \in S$ let $j_t : S \rightarrow S \times I; p \mapsto (p, t)$, and $j^p : I \rightarrow S \times I; t \mapsto (p, t)$. Let $Z \in \mathcal{X}(S \times I)$ be the vector field $(p, t) \mapsto j_*^p(\partial/\partial t)(t)$. For each $\phi \in F^k(S \times I)$, define

$$\mathcal{I}\phi_p = \int_0^1 (j_t^* \phi)_p dt.$$

Since $t \mapsto (j_t^* \phi)_p \in \wedge^k(T_p S)^*$ is continuous, the integral converges. Using the compactness of I , it is easy to show that for every $p \in S$ and $\varphi \in \mathcal{U}_S$ with $p \in U_\varphi$ there are a neighborhood V of φp and a local representative θ of ϕ relative to $\varphi \times \text{Id}$ defined in $V \times I$. Then $\mathcal{I}\theta \in F^k(V)$ is a local representative of $\mathcal{I}\phi$ relative to φ , i.e., $\mathcal{I}\phi \in F^k(S)$. Since $\mathcal{I}d\theta = d\mathcal{I}\theta$, it follows that $\mathcal{I}m_{R \times I} \subseteq m_S$. Thus \mathcal{I} induces a linear map $A^k(S \times I) \rightarrow A^k(S)$, also denoted \mathcal{I} , which commutes with d . Define $M : A^k(S \times I) \rightarrow A^{k-1}(S)$, $k \in \mathbb{Z}$, by

$$(2.12) \quad M\omega = \mathcal{I}i_Z\omega .$$

For each $p \in S$, $k \in \mathbf{Z}$,

$$(j_1^*\omega - j_0^*\omega)_p = \int_I \frac{\partial}{\partial t} ((j_t^*\omega)_p) dt$$

Because Z has local flows on $S \times (0, 1)$ (cf. [7]), we have

$$(2.13) \quad \int_I \frac{\partial}{\partial t} ((j_t^*\omega)_p) dt = \int_I (j_t^* \mathcal{L}_Z \omega)_p dt = \int_I (j_t^*(di_Z\omega + i_Z d\omega))_p dt ,$$

i.e.,

$$(2.14) \quad j_1^*\omega - j_0^*\omega = \mathcal{I}di_Z\omega + \mathcal{I}i_Zd\omega = dM\omega + Md\omega .$$

If $\text{supp } \omega \subseteq U \times I$ for $U \subseteq S$, then $\text{supp } M\omega \subseteq U$. We define a graded presheaf (P, ρ) of R -vector spaces on S by setting

$$P^k(U) = \Gamma(U \times I, \mathcal{X}^k(U \times I, (R \cap U) \times I))$$

and $\rho_{U'}^U = \iota^*$, where $\iota: U' \hookrightarrow U$. Then $M \circ \rho_{U'}^U = \rho_{U'}^U \circ M$. We may thus consider M as a homomorphism from P^k to the presheaf of local sections of $\mathcal{X}^{k-1}(S, R)$.

We also define a presheaf (V, r) of R -vector spaces on S by setting

$$V(U) = \Gamma(U \times I, \mathcal{S} \times I)$$

and letting $r_{U'}^U$ be the ordinary restriction mapping. For each $v \in V(U)$, there is a unique $\sigma \in \Gamma(U, \mathcal{S})$ such that $v = \sigma \times \text{Id}_I$. Thus we have an isomorphism of presheaves

$$\beta_U: V(U) \rightarrow \Gamma(U, \mathcal{S}); \quad \sigma \times \text{Id}_I \mapsto \sigma .$$

Now let $\zeta \in \Gamma(U \times I, \mathcal{X}^k(S, R; \mathcal{S}) \times I)$ for some open $U \subseteq S$, and let $p \in U$. For each $t \in I$ there exist $\varepsilon_t > 0$ and a neighborhood $U_t \subseteq U$ of p such that for $V_t := U_t \times (t - \varepsilon_t, t + \varepsilon_t)$

$$\zeta|_{V_t} \in \Gamma(V_t, \mathcal{X}^k(S, R) \times I) \otimes_R \Gamma(V_t, \mathcal{S} \times I) .$$

Using the compactness of I we can find a neighborhood $W \subseteq U$ of p and a finite partition t_0, \dots, t_{n+1} of I such that the cover $\{V_i := W \times [t_{i-1}, t_{i+1}] \mid i = 1, \dots, n\}$ is a refinement of $\{V_t \mid t \in I\}$. Let $\{w_\alpha\}$ be a basis of the R -vector space $\Gamma(W, \mathcal{S})$. Then for each α and each i there is a unique $\omega^{\alpha, i} \in \Gamma(V_i, \mathcal{X}^k(S, R))$ such that

$$\zeta|_{V_i} = \sum_\alpha \omega^{\alpha, i} \otimes (w_\alpha \times \text{Id}_I) .$$

It follows that $\omega^{\alpha, i-1}$ and $\omega^{\alpha, i}$ agree on their common domain, thus giving rise to sections $\omega^\alpha \in P^k(W)$ such that

$$(2.15) \quad \zeta|_{W \times I} = \sum_{\alpha} \omega^\alpha \otimes (w_\alpha \times \text{Id}_I) .$$

We have thus shown that for each such ζ and p there exists a $W \ni p$ such that $\zeta|_{W \times I} \in P^k(W) \otimes V(W)$.

We now define

$$(2.16) \quad \kappa = (M \otimes \beta) \circ h^* : \Gamma(S', \mathcal{K}^k(S', R'; \mathcal{S}')) \rightarrow \Gamma(S, \mathcal{K}^{k-1}(S, R; \mathcal{S})) .$$

Because f is proper with respect to $\Sigma \times I$ and Σ' , κ is proper with respect to Σ and Σ' . From (2.14), (2.15) and (2.16) it follows that

$$\kappa \circ \delta + \delta \circ \kappa = j_1^* \circ h^* - j_0^* \circ h^* = h_1^* - h_0^* .$$

Therefore we have

$$h_0^* = h_1^* : H_{\Sigma'}^k(S', R'; \mathcal{S}') \rightarrow H_{\Sigma}^k(S, R; \mathcal{S}) , \quad k \in \mathbf{Z} . \quad \text{q.e.d.}$$

Let \mathcal{T} be the category whose objects are triples (S, R, Σ) , S a C^∞ -subcartesian space, $R \subseteq S$ closed, and Σ paracompactifying, and whose morphisms are proper C^∞ -mappings of pairs. Then \mathcal{T} is an admissible category in the sense of Eilenberg-Steenrod. Let V be an R -vector space. From the results of this section it follows that the functors

$$F^k(V) : (S, R, \Sigma) \mapsto H_{\Sigma}^k(S, R; S \times V) , \quad k \in \mathbf{Z} ,$$

form a cohomology theory on \mathcal{T} in the sense of Eilenberg-Steenrod. Moreover, if \mathcal{T}' is an admissible subcategory of \mathcal{T} (e.g., the full subcategory of locally compact pairs and compact supports) then $\{F^k(V)|_{\mathcal{T}'} | k \in \mathbf{Z}\}$ also satisfies the Eilenberg-Steenrod axioms. We therefore have for each of these cohomology theories the well-known series of theorems valid for Eilenberg-Steenrod cohomology theories on admissible categories, including the standard theorems on triads and triples, and the Mayer-Vietories theorems (cf. [4, Chapter 1]).

3. Comparison of the de Rham and sheaf cohomology theories

We begin with an example showing that the de Rham and sheaf cohomology theories ([5] or [3]) are distinct even on the category of finite dimensional compact spaces. We denote the sheaf cohomology functors by H^m .

Example 3.1. Let $S = \{1/n | n \in \mathbf{N}\} \cup \{0\}$ have the C^∞ -structure induced from \mathbf{R} . Set $R = \emptyset$, $\Sigma = \text{cls}$, and let \mathcal{S} be the constant sheaf of real numbers. Then $H^0(S)$ is the direct sum of countably many copies of \mathbf{R} . To compute $H^0(S)$, note that $\dim T_{1/n}S = 0$, $n \in \mathbf{N}$, and $\dim T_0S = 1$. Then $f \in \mathcal{A}^0(S)$ is a zero-

cocycle if and only if $df = 0$, or equivalently, if and only if f has a C^∞ -extension F near 0 satisfying $F'(0) = 0$. If $C^\infty[-1, 1]$ has the C^∞ -topology, then $n := \{f \in C^\infty[-1, 1] \mid f|_S = 0\}$ is a closed subspace, and the projection $C^\infty[-1, 1] \rightarrow C^\infty[-1, 1]/n$ is continuous. Clearly $A^0(S) \approx C^\infty[-1, 1]/n$, and because $C^\infty[-1, 1]$ has a complete linear metric, so does $A^0(S)$. The map $C^\infty[-1, 1] \rightarrow \mathbf{R}; F \mapsto F'(0)$ is continuous and annihilates n . Thus $\{F + n \mid F'(0) = 0\} \cong H^0(S)$ is a closed subspace of $C^\infty[-1, 1]/n$, and hence carries a complete linear metric. On the other hand, the Baire category theorem implies that $H^0(S)$ cannot carry a complete linear metric. Thus $A^0(S)$ and $H^0(S)$ are not isomorphic. This example further shows that H is not continuous in the sense that $\varinjlim H(S, S_m) \not\cong H(S)$, where $S_m = \{1/n \mid n \geq m\}$.

Although \underline{H} and H are not isomorphic, there are spectral sequences relating them. To each $(S, R, \mathcal{S}, \Sigma) \in \mathcal{Q}$ there corresponds the first quadrant double complex

$$C^{k,m}(S, R, \mathcal{S}, \Sigma) := \Gamma_{\Sigma} \mathcal{C}^k(S; \mathcal{K}^m(S, R; \mathcal{S})),$$

where (\mathcal{C}, d) is the canonical resolution of Godement. The first differential d' of the double complex is d , and the second differential d'' is that induced by

$$(-1)^k \delta: \mathcal{K}^m(S, R; \mathcal{S}) \rightarrow \mathcal{K}^{m+1}(S, R; \mathcal{S}).$$

The exact sequence of sheaves

$$(3.2) \quad 0 \rightarrow \mathcal{K}(S, R; \mathcal{S}) \rightarrow \mathcal{A}_S \otimes \mathcal{S} \rightarrow i(\mathcal{A}_R \otimes \mathcal{S}|_R)$$

induces the exact sequence of double complexes

$$0 \rightarrow C(S, R, \mathcal{S}, \Sigma) \rightarrow C(S, \emptyset, \mathcal{S}, \Sigma) \rightarrow C_{\Sigma}(S; i(\mathcal{A}_R \otimes \mathcal{S}|_R)),$$

and this exact sequence is natural with respect to morphisms of \mathcal{Q} . Composing with the restriction mapping

$$C_{\Sigma}(S; i(\mathcal{A}_R \otimes \mathcal{S}|_R)) \rightarrow C(R, \emptyset, \mathcal{S}|_R, \Sigma \cap R)$$

we obtain the (non-exact) sequence

$$(3.3) \quad C(S, R, \mathcal{S}, \Sigma) \rightarrow C(S, \emptyset, \mathcal{S}, \Sigma) \rightarrow C(R, \emptyset, \mathcal{S}|_R, \Sigma \cap R).$$

With each pair $(S, R, \mathcal{S}, \Sigma), (S', R', \mathcal{S}', \Sigma') \in \mathcal{Q}$ there are associated natural homomorphisms

$$(3.4a) \quad C^{k,m}(S, R, \mathcal{S}, \Sigma) \otimes C^{l,n}(S', R', \mathcal{S}', \Sigma') \rightarrow \Gamma_{\Sigma \cap \Sigma'} \mathcal{C}^{k+l}(S; \mathcal{K}^m(S, R; \mathcal{S}) \otimes \mathcal{K}^n(S', R'; \mathcal{S}'))$$

$$(3.4b) \quad \rightarrow C^{k+l, m+n}(S, R \cup R', \mathcal{S} \otimes \mathcal{S}', \Sigma \cap \Sigma'),$$

where the first is the canonical product of Godement [5], and the second is that induced by (1.6).

Definition 3.5. Let (S, R) be a pair of C^∞ -subcartesian spaces, and let \mathcal{S} be a sheaf of R -vector spaces over S . Define $\mathcal{H}^k(S, R; \mathcal{S})$ to be the k -th derived sheaf of $\mathcal{H}(S, R; \mathcal{S})$. Writing $\mathcal{H}(S, R; \mathbf{R}) =: \mathcal{H}(S, R)$ we note that $\mathcal{H}(S, R; \mathcal{S}) = \mathcal{H}(S, R) \otimes \mathcal{S}$ when $R \subseteq S$ is closed. We call $\mathcal{H}(S, R; \mathcal{S})$ (respectively $\mathcal{H}(S, R)$) *the de Rham sheaf of $(S, R; \mathcal{S})$* (respectively (S, R)).

Let Tot be the total complex of C . Then there are the usual two spectral sequences $'E$ and $''E$ satisfying

$$(3.6a) \quad 'E_2^{k,m} = H_{\Sigma}^k(S; \mathcal{H}^m \mathcal{H}(S, R; \mathcal{S})) \Rightarrow H^{k+m} \text{Tot}(S, R, \mathcal{S}, \Sigma),$$

$$(3.6b) \quad ''E_2^{k,m} = H^k H_{\Sigma}^m(S; \mathcal{H}(S, R; \mathcal{S})) \Rightarrow H^{k+m} \text{Tot}(S, R, \mathcal{S}, \Sigma).$$

The edge terms $'E_2^{k,0}$ and $''E_2^{k,0}$ are $H_{\Sigma}^k(S; \mathcal{H}^0(S, R; \mathcal{S}))$ and $H_{\Sigma}^k(S, R; \mathcal{S})$, respectively. The edge homomorphisms

$$(3.7) \quad 'E_2^{k,0} \xrightarrow{\alpha} H^k \text{Tot}(S, R, \mathcal{S}, \Sigma) \xleftarrow{\beta} ''E_2^{k,0}$$

are induced by the chain maps:

$$(3.8) \quad \begin{array}{ccc} C_{\Sigma}(S; \mathcal{H}^0(S, R; \mathcal{S})) & & \\ \downarrow \eta & & \\ C(S, R, \mathcal{S}, \Sigma) & \longrightarrow & \text{Tot}(S, R, \mathcal{S}, \Sigma) \\ \uparrow \omega & & \\ K_{\Sigma}(S, R; \mathcal{S}) & & \end{array}$$

It follows that the edge homomorphisms are natural with respect to morphisms of \mathcal{Q} , and they respect cup products.

Theorem 3.9. *If Σ is paracompactifying, then β is an isomorphism. If Σ is paracompactifying for the pair (S, R) , then $\beta^{-1} \circ \alpha$ is natural with respect to all connecting maps b^k (cf. Theorem 2.1). If R is closed, then $\beta^{-1} \circ \alpha$ is natural with respect to the connecting maps Δ^k .*

Proof. If Σ is paracompactifying, then Lemma 1.2 implies $H_{\Sigma}^m(S; \mathcal{H}(S, R; \mathcal{S})) = 0$ for $m \neq 0$. Thus $''E_2^{k,m} = 0$ for $m \neq 0$, and it follows that β is an isomorphism (cf. [3, Chapter IV]). A short exact sequence of coefficient sheaves induces a short exact sequence of the corresponding double complexes, i.e., a short exact sequence of diagrams (3.8). If Σ is paracompactifying for the pair (S, R) , then these give long exact sequences in cohomology with α and β being natural with respect to connecting maps. If R is closed, then (3.3) is exact and remains exact when augmented on the right by zero. It then follows as usual that α and β are natural with respect to the connecting maps. q.e.d.

Thus when Σ is paracompactifying, (3.6a) gives

$$(3.13) \quad \begin{aligned} {}^1E_2^{k,m} = H_\Sigma^k(S; \mathcal{K}^m(S, R; \mathcal{S})) &\Rightarrow H^{k+m}H_\Sigma^0(S; \mathcal{K}(S, R; \mathcal{S})) \\ &\approx H_\Sigma^{k+m}(S, R; \mathcal{S}) . \end{aligned}$$

Theorem 3.11. *Suppose $\mathcal{K}^k(S, R; \mathcal{S}) = 0$ for $k > 0$. Then α is an isomorphism, and there results the natural isomorphism of cohomology algebras*

$$(3.12a) \quad H_\Sigma(S, R; \mathcal{S}) \approx H_\Sigma(S; \mathcal{H}^0(S, R; \mathcal{S})) .$$

If Σ is paracompactifying for the pair (S, R) , then (3.12a) is natural with respect to the connecting maps b^k . If R is closed, then

$$(3.12b) \quad H_\Sigma(S, R; \mathcal{S}) \approx H_\Sigma(S, R; \mathcal{H}^0(S; \mathcal{S})) ,$$

and (3.12b) is natural with respect to the connecting maps Δ^k .

Proof. The proofs of (3.12a) and the naturality of the b^k are standard ([3, IV. 2] or [5, § 4]). If R is closed, then $\mathcal{H}^0(S, R; \mathcal{S}) = \mathcal{H}^0(S; \mathcal{S})_{S \setminus R}$ and $H_\Sigma(S, R; \mathcal{H}^0(S; \mathcal{S})) \approx H_\Sigma(S; \mathcal{H}^0(S, R; \mathcal{S}))$ [3, Proposition II. 12.2] from which (3.12b) and the naturality of the Δ^k follow.

Corollary 3.13 (*de Rham isomorphism*). *If Σ is paracompactifying, $(S, R, \mathcal{S}, \Sigma)$ is locally contractible (in \mathcal{Q}), and $R \subseteq S$ is closed, then*

$$(3.14) \quad H_\Sigma(S, R; \mathcal{S}) \approx H_\Sigma(S, R; \mathcal{S}) .$$

This isomorphism is natural with respect to morphisms of \mathcal{Q} , and is also natural with respect to connecting maps.

Proof. Theorems 2.9 and 2.11 imply

$$\mathcal{H}^k(S, R; \mathcal{S})_p \approx \begin{cases} \mathcal{S}_p , & \text{for } p \notin R \text{ and } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$H_\Sigma(S, R; \mathcal{S}) \approx H_\Sigma(S; \mathcal{H}^0(S, R; \mathcal{S})) \approx H_\Sigma(S, R; \mathcal{S}) .$$

Example 3.15. Let $f: R \rightarrow R$ be continuous but nowhere differentiable, and define $f_1(t) = \int_0^t f$. Let S be the graph of f_1 equipped with the structure induced from R^2 . For each $p \in S$, $\dim T_p S = 2$. S is nowhere locally C^∞ -contractible. Let $g \in F^0(S)$ satisfy $dg = 0$. Then for $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2) \in S$, $g(p_2) - g(p_1) = \int_{x_1}^{x_2} (g \circ f_1)' dt = 0$. Thus $\mathcal{H}^0 := \mathcal{H}^0(S, \emptyset; R)$ is the constant sheaf of real numbers, and $H^0(S) \cong R$. Clearly $H^m(S) = 0$ for $m > 2$. If $\omega \in F^2(S)$, and $\Omega = g dx \wedge dy$ is a local representative of ω , then Ω is defined

in some open convex neighborhood U of S in \mathbf{R}^2 . It follows from the classical Poincaré lemma that Ω is exact in U . Hence ω is exact, $\mathcal{H}^2 = 0$, and $H^2(S) = 0$. To compute $H^1(S)$, let $\omega \in F^1(S)$ be closed, and let $\Omega = g dx + h dy$ be a local representative of ω defined in U . Then $d\Omega(x, f_1(x)) = 0$ for each $x \in \mathbf{R}$, i.e.,

$$(3.16) \quad \frac{\partial g}{\partial y} \Big|_S = \frac{\partial h}{\partial x} \Big|_S.$$

For each $(x, y) \in U$ define

$$G(x, y) = g(x, f_1(x)) + \int_{f_1(x)}^y \frac{\partial h}{\partial x}(x, t) dt.$$

Then $\partial G/\partial y = \partial h/\partial x$ in U . Using (3.16) one easily shows that $G \in C^\infty(U)$. Thus $\Theta = Gdx + hdy \in F^1(U)$, and Θ is closed and hence exact. Finally, $\Theta|_S = \Omega|_S = \omega$. Thus ω is exact. We conclude that $\mathcal{H}^1 = 0$ and $H^1(S) = 0$; hence $H(S) \approx H(S)$.

We now compare $H(S)$ with the cohomology of S as introduced by Smith [11]. Let $\mathcal{F} = F^0(S)$, and let $\mathfrak{S}(S, \mathcal{F})$ denote the Smith cohomology of the pair (S, \mathcal{F}) . If $V \subseteq \mathbf{R}^n$ is open and $g: V \rightarrow S$ is continuous such that $f \circ g \in C^\infty(V)$ for every $f \in \mathcal{F}$, then certainly both $\pi_x \circ g$ and $\pi_y \circ g$ are of class C^∞ , where $\pi_x: (x, y) \mapsto x$ and $\pi_y: (x, y) \mapsto y$. Since $f_1 \circ \pi_x \circ g = \pi_y \circ g \in C^\infty(V)$, $\pi_x \circ g$ and hence g are constant maps. The Smith completion \mathcal{F}^* of \mathcal{F} is then $C(S)$, all continuous \mathbf{R} -valued functions on S , and each is a 0-cocycle in the Smith theory. Thus $\mathfrak{S}^0(S, \mathcal{F}) = C(S)$.

4. Appendix: The C^k -de Rham cohomology theory

Throughout let $0 \leq k \leq l \leq \infty$. If S is a paracompact C^l -subcartesian space, then S admits C^k -partitions of unity. If S is of class C^{l+1} , then TS is of class C^l . If S is of class C^{l+2} , then the Lie product of two C^{k+1} vector fields is defined and is of class C^k .

Let $C^k F^m(S)$ denote the m -forms on S of class C^k , and let ${}^k F^m(S) \subseteq C^k F^m(S)$ denote the m -forms on S having a differential also of class C^k (e.g., closed m -forms). Let ${}^k m^n(S)$ be the differentials of 0 in ${}^k F^m(S)$, and define ${}^k A^m(S) = {}^k F^m(S) / {}^k m^m(S)$.

Let (S, R) be a pair of C^{l+2} -subcartesian spaces. Let ${}^{l+2} \mathcal{Q}$, ${}^{k+1} \mathcal{K}^m(S, R; \mathcal{S})$, and ${}^{k+1} K_x^m(S, R; \mathcal{S})$ be the obvious analogies of \mathcal{Q} , $\mathcal{K}^m(S, R; \mathcal{S})$ and $K_x^m(S, R; \mathcal{S})$. Let ${}^{k+1} H_x(S, R; \mathcal{S})$ be the homology of the complex $({}^{k+1} K_x(S, R; \mathcal{S}), \delta)$. Then ${}^{k+1} H$ is a connected family of functors on ${}^{l+2} \mathcal{Q}$ as before, satisfying the excision and dimension axioms.

To check the homotopy axiom, let $h: S \times I \rightarrow S'$ be a homotopy of class C^{k+2} . If $\omega \in {}^{k+1} F^m(S')$, then $h^* \omega \in {}^{k+1} F^m(S \times I)$. If ω is closed, then so is $h^* \omega$, and $di_\gamma h^* \omega = \mathcal{L}_\gamma h^* \omega$, where \mathcal{L}_γ is as in Theorem 2.11. Thus

$$Mh^*\omega = \int_0^1 (j_t^* \mathcal{L}_Z h^*\omega) dt ,$$

and this is evidently an element of $C^{k+1}F^m(S)$. Since

$$d \circ M + M \circ d = j_1^* - j_0^* ,$$

it follows that $dMh^*\omega \in C^{k+1}F(S)$. Thus $Mh^*\omega \in {}^{k+1}F(S)$. The homotopy axiom now follows as before.

The results of § 3 remain valid in the C^k -case. Thus if ${}^{k+1}\mathcal{H}^m(S, R; \mathcal{S}) = 0$ for $m > 0$, then

$${}^{k+1}H_{\Sigma}(S, R; \mathcal{S}) \approx H_{\Sigma}(S; {}^{k+1}\mathcal{H}^0(S, R; \mathcal{S}))$$

when Σ is paracompactifying. In particular, if $(S, R, \mathcal{S}, \Sigma) \in {}^{\iota+2}\mathcal{Q}$ is C^{k+2} -locally contractible, then

$${}^{k+1}H_{\Sigma}(S, R; \mathcal{S}) \approx H_{\Sigma}(S, R; \mathcal{S}) .$$

We end by showing that kH is not a topological invariant for $k < \infty$.

Example. Let S be an arc in \mathbf{R}^{k+2} for which there is a function $f \in C^{k+1}(\mathbf{R}^{k+2})$ with $df|_S = 0$ but $f|_S$ not constant (cf. [14]). Then ${}^{k+1}H^0(S) \supseteq \mathbf{R}$. On the other hand, ${}^{k+1}H^0(\mathbf{R}) = \mathbf{R}$.

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